THE UNIVERSITY OF ALBERTA

SOME ASYMPTOTIC NON-CENTRAL DISTRIBUTIONS OF THE SAMPLE LINEAR REGRESSION COEFFICIENT

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled 'Some Asymptotic Non-Central Distributions of the Sample Linear Regression Coefficient', submitted by Peter Capell in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The purpose of this thesis is to find the approximate distribution of sample regression coefficients based on samples from two linear time series. Chapter I introduces the problem. In Chapter II the distribution of the regression coefficients with known and unknown means, when sampling from a bivariate normal population with zero correlation, are derived, using Daniels's (1956) form of Geary's (1944) extension of Cramér's theorem. In Chapter III Daniels's method is again used, together with the saddle-point approximation and the method of steepest descents to find an asymptotic approximation of the distributions of the same coefficients as in Chapter II , when sampling from two linear time series.

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CHAPTER I

INTRODUCTION

In regression and correlation analysis of bivariate data, a common error is to infer the existence of a direct relationship between the two variables, whereas, in reality, both are related to a third, hidden variable.

Let (x_1, y_1), (x_2, y_2), . . , (x_n, y_n), be a sample of size n from a bivariate population. Denote the sample regression coefficient of y on x by

$$b_{21}^{x} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \tag{1.1}$$

and the sample regression coefficient with known means by

$$b_{21} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} \quad . \tag{1.2}$$

The distribution of b_{21}^{x}, when sampling from a bivariate normal population, was first derived by K. Pearson (1926) and Romanovsky (1926).

In Chapter II of this thesis we derive the corresponding

distributions of b_{21} and b_{21}^{x} in the case where the population has zero correlation, using Daniels's (1956) form of Geary's (1944) extension of Cramér's theorem.

In Chapter III, section I, we derive the approximate distribution of b_{21} when sampling at times $t_1, t_2, .., t_n$, from two linear time series, passing through the origin, with normal residuals. Thus we have

$$
\begin{aligned}
x_i &= B_1\, t_i + e_i, \\
y_i &= B_2\, t_i + f_i, \text{ for } i = 1, 2, .., n,
\end{aligned}
\tag{1.3}
$$

where B_1 and B_2 are constants, and $\{e_i, f_i\}$ are independent N(0, 1) random variables. We again use Daniels's method to find an asymptotic approximation of the inversion integral (2. 1) involved in using the saddlepoint approximation and the method of steepest descents (see De Bruijn (1961)). In section II the same procedure is followed to derive the approximate distribution of b_{21}^{x} when sampling at times $t_1, t_2, .., t_n$, from the series

$$
\begin{aligned}
x_i &= \alpha_1 + B_1\, t_i + e_i, \\
y_i &= \alpha_2 + B_2\, t_i + f_i, \text{ for } i = 1, 2, .., n,
\end{aligned}
\tag{1.4}
$$

where α_1 and α_2 are constants and B_1, B_2, and $\{e_i, f_i\}$ are as defined for (1. 3). The corresponding distributions of b_{12} and b_{21}^{x}, the coefficients of regression of x on y, can be found by permutating the indices in the distributions of b_{21} and b_{12}^{x} respectively.

Daniels's method is given in Appendix I.

The distribution of b_{21} is tabulated for various exemplary values of B_1, B_2, and n, in Appendix II.

CHAPTER II

THE DISTRIBUTION OF THE SAMPLE REGRESSION COEFFICIENT

If r is a statistic of the form c/c_o, where c_o is non-negative, the probability density function (p. d. f.) of r is given by

$$h(r) = \frac{1}{2\pi i}\int \frac{\partial M}{\partial u}(u-rT, T)\Big|_{u=0} \, dT , \tag{2.1}$$

where $M(T_o, T) = E\, e^{T_o c_o + Tc}$, is the joint moment-generating function of c_o and c. Integration is along the imaginary axis in the complex T plane, or any allowable deformation. This is Daniels's form of Geary's extension of Cramér's theorem.

Let (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n), be n observed pairs of values from a bivariate normal distribution with zero correlation. The joint distribution of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$, is

$$dF = (2\pi)^{-n} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2\right]\right\} \tag{2.2}$$

$$\times \quad dx_1 \ldots dx_n \, dy_1 \ldots dy_n \, .$$

Let

$$c = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}), \text{ and}$$

(2. 3)

$$c_o = \sum_{i=1}^{n} (x_i - \bar{x})^2 ,$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i , \text{ and}$$

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i .$$

The joint moment-generating function of c_o and c is

$$M(T_o, T) = (2\pi)^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\left[\sum_{i=1}^{n} x_i^2 \right.\right.$$

$$\left.\left. + \sum_{i=1}^{n} y_i^2 - 2 T_o \sum_{i=1}^{n} (x_i - \bar{x})^2 - 2 T \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \right]\right\}$$

$$dx_1 \ldots dx_n \, dy_1 \ldots dy_n .$$

We now replace $x_1, \ldots, x_n$ by new variables $w_1, \ldots, w_n$ by means of an orthogonal transformation such that

$$w_1 = \sqrt{n}\, \bar{x} , \tag{2.4}$$

and apply a transformation with the same matrix to $y_1, \ldots, y_n$, which are thus replaced by new variables $z_1, \ldots, z_n$, such that

$$z_1 = \sqrt{n}\, \bar{y} .$$

We have then

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=2}^{n} w_i^2 ,$$

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2 ,$$

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} x_i^2 - n\,\bar{x}^2 = \sum_{i=2}^{n} w_i^2 ,$$

and

$$\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \sum_{i=1}^{n} x_i y_i - n\,\bar{x}\,\bar{y} = \sum_{i=2}^{n} w_i z_i$$

Hence

$$M(T_o, T) = (2\pi)^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\left[w_1^2 + z_1^2 \right.\right.$$

$$+ (1 - 2\,T_o) \sum_{i=2}^{n} w_i^2 - 2\,T \sum_{i=2}^{n} w_i z_i$$

$$\left.\left. + \sum_{i=2}^{n} z_i^2 \right]\right\} \, d w_1 \ldots dw_n \, dz_1 \ldots dz_n .$$

This is the product of n double integrals.

Thus

$$M(T_o, T) = \left|\underline{A}\right|^{-\frac{(n-1)}{2}} ,$$

where

$$\underline{A} = \begin{Vmatrix} 1 - 2T_o & -T \\ -T & 1 \end{Vmatrix} .$$

Hence

$$M(T_o, T) = (1 - 2T_o - T^2)^{-\frac{(n-1)}{2}} . \qquad (2.5)$$

Making the substitution

$$u = T_o + b_{21}^{x} T ,$$

we obtain

$$M(u - b_{21}^{x} T, T) = (1 - 2u + 2\, b_{21}^{x} T - T^2)^{-\frac{(n-1)}{2}} .$$

Differentiating with respect to u, and putting u = 0, we obtain,

$$\frac{\partial M}{\partial u} (u - b_{21}^{x} T, T) \Bigg|_{u=0}$$

$$= (n-1)\ (1 + 2\, b_{21}^{x}\ T - T^2)^{-\frac{(n+1)}{2}} .$$

Thus from (2.1), the p.d.f. of b_{21}^{x} is given by

$$h(b_{21}^{x}) = \frac{(n-1)}{2\pi i} \int (1 + 2\, b_{21}^{x} T - T^2)^{-\frac{(n+1)}{2}}\ dT . \qquad (2.6)$$

The integrand $\phi(T) = (1 + 2\, b_{21}^{x} T - T^2)^{-\frac{(n+1)}{2}}$ has two singularities at

the points

$$T = b_{21}^{x} \pm \sqrt{1 + b_{21}^{x\,2}} \ .$$

For any real value of b_{21}^{x} these two points are real and are located on either side of the origin and of the point $(b_{21}^{x}, 0)$ in the complex T plane. Thus we can deform the path of integration from the imaginary axis in the complex T plane to the parallel path through the point $(b_{21}^{x}, 0)$ on the real axis. The value of the integral taken along this new path will be unaltered since

$$\lim_{\text{real } t \to \infty} \left[\int_{(0,-it)}^{(b_{21}^{x},-it)} \phi(T)dT + \int_{(b_{21}^{x},it)}^{(0,it)} \phi(T)dT \right] = 0 \ , \qquad (2.7)$$

where integration is along the straight lines joining the points indicated.

Thus letting $T = b_{21}^{x} + i\eta$ in (2.6) we have

$$h(b_{21}^{x}) = \frac{(n-1)}{\pi i} \int_{0}^{\infty} \left[1 + b_{21}^{x\,2} + \eta^{2} \right]^{-\frac{(n+1)}{2}} i d\eta \ . \qquad (2.8)$$

Let

$$\xi = \left[1 + \frac{\eta^{2}}{(1+b_{21}^{x\,2})} \right]^{-1} ,$$

so that

$$d\eta = - \frac{(1 + b_{21}^{x\,2})^{\frac{1}{2}}}{2\xi^{\frac{3}{2}} (1-\xi)^{\frac{1}{2}}} \ d\xi \ .$$

[illegible]

Thus

$$h(b_{21}^{x}) = \frac{(n-1)}{2\pi} \left(1 + b_{21}^{x\,2}\right)^{-\frac{n}{2}} \int_{0}^{1} \xi^{\frac{n}{2}-1} (1-\xi)^{-\frac{1}{2}} \, d\xi , \tag{2.9}$$

which reduces to

$$h(b_{21}^{x}) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\;\Gamma(\frac{n-1}{2})} \left(1 + b_{21}^{x\,2}\right)^{-\frac{n}{2}} . \tag{2.10}$$

If we define the new variable

$$t^{x} = \sqrt{n-1}\;\; b_{21}^{x} ,$$

it can be seen that the p.d.f. of t^{x} is given by

$$p(t^{x}) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi(n-1)}\;\Gamma(\frac{n-1}{2})} \left(1 + \frac{t^{x\,2}}{n-1}\right)^{-\frac{n}{2}} . \tag{2.11}$$

This is Student's 't' distribution with n-1 degrees of freedom. If we replace (2.3) by

$$c = \sum_{i=1}^{n} x_i y_i \text{ and} \tag{2.12}$$

$$c_o = \sum_{i=1}^{n} x_i^{2} ,$$

and follow the same procedure without the orthogonal transformation (2.4) we see that the p.d.f. of b_{21} is given by

$$h(b_{21}) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\;\Gamma(\frac{n}{2})} (1 + b_{21}{}^2)^{-\frac{(n+1)}{2}} \quad . \tag{2.13}$$

Thus the variable

$$t = \sqrt{n}\;\; b_{21}$$

has Student's 't' distribution with n degrees of freedom.

CHAPTER III

THE APPROXIMATE DISTRIBUTION OF THE SAMPLE REGRESSION COEFFICIENT BASED ON SAMPLES FROM TWO LINEAR TIME SERIES

Section 1

In this section we follow the method of the second chapter to find the approximate distribution of

$$b_{21} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

where

$$x_i = B_1 t_i + e_i ,$$
$$y_i = B_2 t_i + f_i, \text{ for } i = 1, 2, \ldots, n , \tag{3.1.1}$$

where B_1 and B_2 are constants and $\{e_i, f_i\}$ are independent $N(0, 1)$ random variables.

We consider the case

$$\sum_{i=1}^{n} t_i^2 = O(n). \tag{3.1.2}$$

If, for example, we choose a fixed time interval of length K and take

the sample values at n equally spaced time throughout this interval, we have

$$t_i = K \frac{i}{n} \text{ for } i = 1, 2, \ldots, n.$$

This gives $\sum_{i=1}^{n} t_i^2 = \frac{K^2}{6n} (n+1)(2n+1) = O(n)$.

We consider this case to be more interesting than any others, as it would probably be of more practical value.

The joint distribution of $x_1, x_2 \ldots, x_n, y_1, y_2, \ldots, y_n$, as defined in (3. 1. 1) is

$$dF = (2\pi)^{-n} \exp\left\{ -\frac{1}{2}\left[\sum_{i=1}^{n} (x_i - B_1 t_i)^2 \right.\right.$$

$$\left.\left. + \sum_{i=1}^{n} (y_i - B_2 t_i)^2 \right]\right\} dx_1 \ldots dx_n \, dy_1 \ldots dy_n .$$

Let

$$c = \sum_{i=1}^{n} x_i y_i , \quad \text{and}$$

$$c_o = \sum_{i=1}^{n} x_i^2 . \tag{3. 1. 3}$$

The joint moment-generating function of c_o and c is

$$M(T_o, T) = (2\pi)^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\left[\sum_{i=1}^{n} (x_i - B_1 t_i)^2 \right.\right.$$

$$+ \sum_{i=1}^{n} (y_i - B_2 t_i)^2 - 2T_o \sum_{i=1}^{n} x_i^2$$

$$- 2T \sum_{i=1}^{n} x_i y_i \Big]\Big\} dx_1 \ldots dx_n \, dy_1 \ldots dy_n , \tag{3.1.4}$$

$$= (2\pi)^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\Big[(1-2T_o) \sum_{i=1}^{n} x_i^2 \right.$$

$$- 2 B_1 \sum_{i=1}^{n} x_i t_i + \sum_{i=1}^{n} y_i^2 - 2B_2 \sum_{i=1}^{n} y_i t_i$$

$$\left. - 2 T \sum_{i=1}^{n} x_i y_i + (B_1^2 + B_2^2) \sum_{i=1}^{n} t_i^2 \Big] \right\}$$

$$\times \; dx_1 \ldots dx_n \, dy_1 \ldots dy_n .$$

Except for a constant factor, (3.1.4) is the product of n double integrals of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} (x, y) \left[\underline{A} \binom{x}{y} - 2 \binom{a}{b} \right] \right\} dx \, dy , \tag{3.1.5}$$

where $\underline{A}$ is a symmetric 2 x 2 constant matrix, and a and b are constants. We denote the integrand of (3.1.5) by

$$\exp\left\{ -\frac{1}{2} \, \phi(x, y) \right\}, \tag{3.1.6}$$

where $\phi(x, y)$ can be written in the form

$$\phi(x, y) = (x, y) \left[\underline{A} \binom{x}{y} - \binom{a}{b} \right]$$

$$+ (a, b) \left[\underline{A}^{-1} \binom{a}{b} - \binom{x}{y} \right] - (a, b) \underline{A}^{-1} \binom{a}{b} \tag{3.1.7}$$

or

$$\begin{aligned} \phi(x, y) = & \left[(x, y) - (a, b) \underline{A}^{-1} \right] \underline{A} \left[\binom{x}{y} - \underline{A}^{-1} \binom{a}{b} \right] \\ & - (a, b) \underline{A}^{-1} \binom{a}{b} . \end{aligned} \tag{3.1.8}$$

Since $\underline{A}$ is symmetric, we can write (3.1.8) as

$$(w, z) \underline{A} \binom{w}{z} - (a, b) \underline{A}^{-1} \binom{a}{b} ,$$

where

$$(w, z) = (x, y) - (a, b) \underline{A}^{-1} . \tag{3.1.9}$$

Clearly the Jacobian of this transformation is unity, so

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[- \frac{1}{2} \phi(x, y) \right] dx\, dy \tag{3.1.10}$$

$$= \exp \left[\frac{1}{2} (a, b) \underline{A}^{-1} \binom{a}{b} \right]$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ - \frac{1}{2} (w, z) \underline{A} \binom{w}{z} \right\} dw\, dz .$$

$$= \frac{2\pi}{|\underline{A}|^{\frac{1}{2}}} \exp \left\{ \frac{1}{2} (a, b) \underline{A}^{-1} \binom{a}{b} \right\} .$$

Using this result to evaluate (3.1.4) we have

$$M(T_o, T) = (2\pi)^{-n} \exp\left\{-\frac{1}{2} \sum_{i=1}^{n} t_i^2 (B_1^2 + B_2^2)\right\}$$

$$\times \prod_{i=1}^{n} \left[\frac{2\pi}{|\underline{A}|^{\frac{1}{2}}} \exp\left\{\frac{1}{2} (a_i\ b_i)\ \underline{A}^{-1} \binom{a_i}{b_i}\right\}\right], \tag{3.1.11}$$

where

$$\underline{A} = \begin{Vmatrix} 1-2T_o & -T \\ -T & 1 \end{Vmatrix},$$

$a_i = B_1 t_i$, and $b_i = B_2 t_i$,

for $i = 1, 2, \ldots, n$.

Hence

$$\underline{A}^{-1} = (1 - 2T_o - T^2)^{-1} \begin{Vmatrix} 1 & T \\ T & 1-2T_o \end{Vmatrix},$$

and we obtain

$$M(T_o, T) = (1-2T_o-T^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2} \sum_{i=1}^{n} t_i^2 (B_1^2+B_2^2)\right\}$$

$$\times \exp\left\{\frac{\sum_{i=1}^{n} t_i^2}{2(1-2T_o-T^2)} \left[B_1^2+(1-2T_o)B_2^2+2T\, B_1 B_2\right]\right\}. \tag{3.1.12}$$

Putting $m = \sum_{i=1}^{n} t_i^2$ and

$$u = T_o + b_{21} T \text{ ,}$$

we obtain

$$M(u-b_{21}\ T, T) = \frac{e^{-\frac{m}{2}(B_1{}^2+B_2{}^2)}}{(1-2u + 2b_{21}\ T-T^2)^{\frac{n}{2}}}$$

$$\times \exp\left\{\frac{m}{2(1-2u+2b_{21}T-T^2)}\ \left[B_1{}^2 + (1-2u+2b_{21}T)\ B_2{}^2+2TB_1B_2\right]\right\} .$$

(3. 1. 13)

Differentiating with respect to u we have

$$\frac{\partial M}{\partial u}\ (u-b_{21}\ T, T)$$

$$= M(u-b_{21}\ T, T)\ \left[\frac{n}{(1-2u+2b_{21}\ T-T^2)}\right.$$

$$+ \frac{m}{(1-2u+2b_{21}\ T-T^2)^2}\ \left\{-\ B_2{}^2(1-2u+2b_{21}\ T-T^2)\right.$$

$$\left.\left. + \left[B_1{}^2+(1-2u+2b_{21}T)\ B_2{}^2+2T\ B_1B_2\right]\right\}\right] ,$$

(3. 1. 14)

and putting u = 0 we have

$$\frac{\partial M}{\partial u}(u-b_{21}T, T)\Big|_{u=0}$$

$$= \frac{e^{-\frac{m}{2}(B_1^2+B_2^2)}}{(1+2b_{21}T-T^2)^{\frac{n}{2}+2}}\left[n(1+2b_{21}T-T^2)+m(B_2T+B_1)^2\right]$$

$$\times \exp\left\{\frac{m}{2}\left[\frac{B_1^2+B_2^2+2TB_2(b_{21}B_2+B_1)}{1+2b_{21}T-T^2}\right]\right\}. \quad (3.1.15)$$

Thus from (2.1) the p.d.f. of b_{21} is given by

$$h(b_{21}) = \frac{e^{-\frac{m}{2}(B_1^2+B_2^2)}}{2\pi i}$$

$$\times \int \frac{\left[n(1+2b_{21}T-T^2)+m(B_2T+B_1)^2\right]}{\left[1+2b_{21}T-T^2\right]^{\frac{n}{2}+2}}$$

$$\times \exp\left\{\frac{m}{2}\left[\frac{B_1^2+B_2^2+2TB_2(b_{21}B_2+B_1)}{1+2b_{21}T-T^2}\right]\right\} dT,$$

(3.1.16)

where integration is along the imaginary axis in the complex T plane, or along any allowable deformation.

In the case (3.1.2), with a possible relative error which is $O(n^{-1})$ we may write $m = Rn$, where R is a constant. In (3.16) where m occurs it is multiplied by homogeneous terms of the second degree in B_1 and B_2 and we may, without loss of generality, take

$$m = n,$$

and write (3.1.16) in the form

$$h(b_{21}) = \frac{n}{2\pi i} e^{-\frac{n}{2}B_1^2} \int \frac{\left[1+2b_{21}T-T^2 + (B_2T+B_1)^2\right]}{\left[1 + 2b_{21} T-T^2\right]^{\frac{n}{2}+2}}$$

$$x \exp\left\{\frac{n}{2}\left[\frac{(B_1+B_2T)^2}{1+2b_{21}T-T^2}\right]\right\} dT . \tag{3.1.17}$$

Let

$$z = \frac{T - b_{21}}{\sqrt{1+b_{21}^2}} , \quad A = \frac{b_{21} B_2+B_1}{\sqrt{1+b_{21}^2}} , \tag{3.1.18}$$

$$p^2 = B_2^2 + A^2 , \quad \text{and } k = B_2A .$$

Then we have

$$(1 + 2 b_{21} T-T^2) = (1 + b_{21}^2) - (b_{21}-T)^2$$

$$= (1 + b_{21}^2) (1 - z^2) ,$$

and

$$(B_2T + B_1)^2 = \left\{B_2(T-b_{21}) + (b_{21}B_2 + B_1)\right\}^2$$

$$= (1 + b_{21}^2) (B_2z + A)^2 .$$

Thus

$$(1 + 2b_{21} T-T^2) + (B_2T + B_1)^2$$

$$= (1 + b_{21}^2) \left[1 + A^2 + 2kz + (B_2^2-1) z^2\right] ,$$

and

$$\frac{(B_2 T + B_1)^2}{(1+2b_{21} T - T^2)} = \frac{B_2{}^2 z^2 + 2kz + A^2}{(1 - z^2)}$$

$$= \frac{p^2 z^2 + 2kz}{(1 - z^2)} + A^2 .$$

Substituting in (3. 1. 17) we have

$$h(b_{21}) = \frac{n}{2\pi i} e^{-\frac{n}{2}(B_1{}^2 - A^2)} (1+b_{21}{}^2)^{-\frac{(n+1)}{2}}$$

$$x \int \frac{\left[1+A^2 + 2kz + (B_2{}^2 - 1)z^2\right]}{(1 - z^2)^2}$$

$$x \exp\left\{\frac{n}{2}\left[\frac{2kz + p^2 z^2}{1 - z^2} - \ln(1 - z^2)\right]\right\} dz , \tag{3. 1. 19}$$

where integration is along the path

$$Re(z) = - \frac{b_{21}}{\sqrt{1+b_{21}{}^2}} , \tag{3. 1. 20}$$

or any allowable deformation. We evaluate this integral using the following saddlepoint approximation and method of steepest descents,

$$\int f(x)\, e^{tg(x)}\, dx$$

$$= \sqrt{2\pi}\ \alpha \left[t\ |g_{11}(\hat{x})|\right]^{-\frac{1}{2}} f(\hat{x})\, e^{tg(\hat{x})}$$

$$x \left\{1 + O(t^{-1})\right\} \qquad (t \to \infty) , \tag{3. 1. 21}$$

where

$$g_{11}(\hat{x}) = \left.\frac{d^2g}{dx^2}\right|_{x=\hat{x}} \tag{3.1.22}$$

$$\hat{x} \text{ is the solution of } \frac{dg}{dx} = 0 \text{ ,}$$

the path of integration in the neighbourhood of the saddlepoint at $\hat{x}$ is the path of steepest descent, which is directed along the axis of the saddlepoint, that is, along the straight line

$$(x-\hat{x})^2 \; g_{11}(\hat{x}) \text{ real and } \leq 0 \text{ ,} \tag{3.1.23}$$

and α is the complex number with modulus 1 whose argument corresponds to the direction of the axis (see, for example, De Bruijn (1961), Chapter 5).

The integral in (3.1.19) is in the form of the left member of (3.1.21) with x replaced by z, where

$$g(z) = \frac{2kz + p^2z^2}{1 - z^2} \quad -\ell n(1-z^2) \text{ ,}$$

$$f(z) = \frac{\left[1 + A^2 + 2kz + (B_2{}^2-1)z^2\right]}{(1 - z^2)^2} \tag{3.1.24}$$

and $t = \frac{n}{2}$.

The only singularities of the integrand of (3.1.19) are at

$$z = \pm 1$$

and, from (3.1.20), for any finite value of b_{21}, the path of integration

in the complex z plane is parallel to the imaginary axis and cuts the real axis at a point in the interval (-1, 1). We shall show that there is always a unique point $\hat{z} \in (-1, 1)$, such that

$$\left.\frac{dg}{dz}\right|_{z=\hat{z}} = 0 .$$

From (3. 1. 24) we have

$$\frac{dg}{dz} = \frac{2}{(1-z^2)^2}\left[k + (p^2+1)\, z + kz^2 - z^3\right] \tag{3. 1. 25}$$

and

$$\frac{d^2g}{dz^2} = \frac{2}{(1-z^2)^2}\left[p^2+1 + 2kz - 3z^2\right] - \frac{4z}{(1-z^2)}\ \frac{dg}{dz} .$$

Thus, from (3. 1. 22),

$$g_{11}(\hat{z}) = \frac{2}{(1-\hat{z}^2)^2}\left[p^2+1 + 2k\hat{z} - 3\hat{z}^2\right] . \tag{3. 1. 26}$$

Let

$$h(z) = k + (p^2+1)z + kz^2 - z^3 .$$

From (3. 1. 25) and (3. 1. 26) we see that for real z, if $z^2 \neq 1$,

$$h(\hat{z}) = 0 \Rightarrow \left.\frac{dg}{dz}\right|_{z=\hat{z}} = 0 \tag{3. 1. 27}$$

and

$$g_{11}(\hat{z}) \text{ has the same sign as } \left.\frac{dh}{dz}\right|_{z=\hat{z}} . \tag{3. 1. 28}$$

Also $h(z)$ is a continuous function of z with three roots and $\frac{dh}{dz}$ is a continuous function of z with two roots. We denote the real parts of the roots of $h(z)$ by $A \leq B \leq C$.

From the definitions (3.1.18) we note

$$p^2 \geq |2k| .$$

(a) Suppose $p^2 \neq \pm 2k$.

Then $p^2 > |2k|$

and we have

$$h(-1) = 2k - p^2 < 0 ,$$

$$h(1) = 2k + p^2 > 0 ,$$

$$h(\infty) = -\infty \quad \text{and}$$

$$h(-\infty) = \infty .$$

Thus $h(z)$ has three real roots where

$$A \in (-\infty, -1), \quad B \in (-1, 1) , \text{ and } C \in (1, \infty).$$

Let $\hat{z} = B$.

(b) Suppose $p^2 = 2k \neq 0$, which implies that $k > 0$. Then

$$h(-1) = 0 , \qquad h(1) = 4k > 0 ,$$

$$h(-\infty) = \infty , \qquad h(\infty) = -\infty , \text{ and}$$

$$\left.\frac{dh}{dz}\right|_{z=-1} = \Big[2k(1+z) + (1-3z^2)\Big]_{z=-1}$$

$$= -2 < 0 .$$

Thus $h(z)$ has three real roots where

$$A = -1, \quad B \in (-1, 1), \quad \text{and } C \in (1, \infty).$$

Let $\hat{z} = B$.

(c) Suppose $p^2 = -2k \neq 0$, which implies that $k < 0$. Then

$h(-1) = 4k < 0$, $\quad$ $h(1) = 0$,

$h(-\infty) = \infty$, $\quad$ $h(\infty) = -\infty$, and

$$\left.\frac{dh}{dz}\right|_{z=1} = \Big[2k(z-1) + (1-3z^2)\Big]_{z=1} = -2 < 0 .$$

Thus h(z) has three real roots where

$A \in (-\infty, -1)$, $\quad$ $B \in (-1, 1)$, and $C = 1$.

Let $\hat{z} = B$.

(d) Suppose $k = 0$. Then

$$h(z) = z(p^2+1 - z^2),$$

so

$h(-1) = -p^2 \leq 0$, $\quad$ $h(0) = 0$, and

$h(1) = p^2 \geq 0$.

Thus h(z) has three real roots where

$A \in (-\infty, -1]$, $\quad$ $B = 0$, and $C \in [1, \infty)$.

Let $\hat{z} = B$.

In all four cases (a), (b), (c), (d), $\frac{dh}{dz}$ must have a root in each of the intervals (A, B) and (B, C), so

$$\left.\frac{dh}{dz}\right|_{z=\hat{z}} \neq 0 ,$$

and since $h(z) < 0$ for $z \in (A, B)$, and $h(z) > 0$ for $z \in (B, C)$, we have

$$\left.\frac{dh}{dz}\right|_{z=\hat{z}} > 0 .$$

Thus from (3. 1. 27) and (3. 1. 28) we see that for any finite values of p^2 and k there exists a unique real number $\hat{z}$ such that

$$\hat{z} \in (-1, 1)\ , \quad \left.\frac{dg}{dz}\right|_{z=\hat{z}} = 0\ , \text{ and } g_{11}(\hat{z}) > 0.$$

From (3. 1. 23) we see that, since $\hat{z}$ is real, and $g_{11}(\hat{z}) > 0$, the axis of the saddlepoint is the straight line

$$Re(z) = \hat{z}\ .$$

Thus, in the neighbourhood of the saddlepoint we can deform the path of integration of (3. 1. 19) to pass along the axis of the saddle-point, and use the saddlepoint approximation (3. 1. 21) with (3. 1. 26), to obtain

$$h(b_{21}) = \sqrt{\frac{n}{2\pi}}\ e^{-\frac{n}{2}(B_1{}^2+B_2{}^2)}\ (1 + b_{21}{}^2)^{-\frac{(n+1)}{2}}$$

$$\times \frac{\left[1 + A^2 + 2k\hat{z} + (B_2{}^2-1)\hat{z}^2\right]}{(1-\hat{z}^2)^{\frac{n}{2}+1}\left[p^2+1+2k\hat{z}-3\hat{z}^2\right]^{\frac{1}{2}}} \quad \exp\left\{\frac{n}{2}\left[\frac{2k\hat{z} + p^2}{(1 - \hat{z}^2)}\right]\right\}$$

$$\times \left\{1 + O(n^{-1})\right\} \quad (n \to \infty). \qquad (3. 1. 29)$$

Putting $B_1 = B_2 = 0$ in definitions (3. 1. 18) we have $A = p^2 = k = 0$, hence from case (d) we have $\hat{z} = 0$. Substituting these values in

(3. 1. 29) we obtain

$$h(b_{21}) = \sqrt{\frac{n}{2\pi}}\,(1 + b_{21}{}^2)^{-\frac{(n+1)}{2}}$$

$$\times \left\{1 + O(n^{-1})\right\} \qquad (n \to \infty)\,. \tag{3. 1. 30}$$

As expected (3. 1. 30) is consistent with (2. 13) which can be obtained exactly from (3. 1. 30) by renormalising.

Section II

In this section we follow the procedure of Section I to find the approximate distribution of

$$b_{21}{}^{x} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})\,(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}\,,$$

where

$$\begin{aligned} x_i &= \alpha_1 + B_1 t_i + e_i\,, \\ y_i &= \alpha_2 + B_2 t_i + f_i\,, \quad \text{for } i = 1, 2, \ldots, n\,, \end{aligned} \tag{3. 2. 1}$$

where $\alpha_1, \alpha_2, B_1, B_2$, are constants, and $\left\{e_i, f_i\right\}$ are independent $N(0, 1)$ random variables.

The joint distribution of $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$, as defined in (3. 2. 1) is

$$dF = (2\pi)^{-n} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}(x_i - \alpha_1 - B_1 t_i)^2 + \sum_{i=1}^{n}(y_i - \alpha_2 - B_2 t_i)^2\right]\right\} dx_1 \ldots dx_n \; dy_1 \ldots dy_n .$$

Let

$$c = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \qquad \text{and}$$

$$c_o = \sum_{i=1}^{n}(x_i - \bar{x})^2 . \tag{3.2.2}$$

The joint moment-generating function of c_o and c is

$$M(T_o, T) = (2\pi)^{-n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}(x_i - \alpha_1 - B_1 t_i)^2 + \sum_{i=1}^{n}(y_i - \alpha_2 - B_2 t_i)^2 \right.\right. \tag{3.2.3}$$

$$\left.\left. - 2T_o \sum_{i=1}^{n}(x_i - \bar{x})^2 - 2T \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right]\right\}$$

$$\times \quad dx_1 \ldots dx_n \; dy_1 \ldots dy_n .$$

We replace $x_1, \ldots, x_n$ by $w_1, \ldots, w_n$ by an orthogonal transformation

$$\begin{Vmatrix} w_1 \\ \vdots \\ w_n \end{Vmatrix} = \underline{B} \begin{Vmatrix} x_1 \\ \vdots \\ x_n \end{Vmatrix} ,$$

where $\underline{B}$ is an n x n orthogonal matrix such that

$$w_1 = \sqrt{n}\ \bar{x} .$$

Thus

$$\underline{B} = \left\Vert \begin{array}{cccc} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots\cdots & \frac{1}{\sqrt{n}} \\ \hline & & \underline{C} & \end{array} \right\Vert \qquad (3.2.4)$$

where $\underline{C}$ is an (n-1) x n matrix. We apply the same transformation to $y_1, \ldots y_n$ to obtain $z_1, \ldots z_n$, where

$$\begin{Vmatrix} z_1 \\ \vdots \\ z_n \end{Vmatrix} = \underline{B} \begin{Vmatrix} y_1 \\ \vdots \\ y_n \end{Vmatrix} ,$$

and thus from the definition of $\underline{B}$ we have

$$z_1 = \sqrt{n}\ \bar{y} .$$

We have then

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} w_i^2 , \qquad \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} z_i^2 , \qquad (3.2.5)$$

$$\sum_{i=1}^{n} (x_i - x)^2 = \sum_{i=2}^{n} w_i^2 , \quad \sum_{i=1}^{n} (x_i - x)(y_i - y) = \sum_{i=2}^{n} w_i z_i ,$$

$$\sum_{i=1}^{n} x_i t_i = (t_1 \ \dots \ t_n) \begin{Vmatrix} x_1 \\ \vdots \\ x_n \end{Vmatrix} = (t_1 \ \dots \ t_n) \underline{B}^{-1} \begin{Vmatrix} w_1 \\ \vdots \\ w_n \end{Vmatrix} ,$$

and $\sum_{i=1}^{n} y_i t_i = (t_1 \ \dots \ t_n) \ \underline{B}^{-1} \begin{Vmatrix} z_1 \\ \vdots \\ z_n \end{Vmatrix}$.

Substituting (3. 2. 5) in (3. 2. 3) we obtain

$$M(T_o, T) = (2\pi)^{-n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left\{ - \frac{1}{2} \left[\sum_{i=1}^{n} w_i^2 + \sum_{i=1}^{n} z_i^2 \right. \right.$$

$$- 2T_o \sum_{i=2}^{n} w_i^2 - 2T \sum_{i=2}^{n} w_i z_i - 2 \alpha_1 \sqrt{n} \, w_1 - 2 \alpha_2 \sqrt{n} \, z_1$$

$$- 2B_1 (t_1 \ \dots \ t_n) \underline{B}^{-1} \begin{Vmatrix} w_1 \\ \vdots \\ w_n \end{Vmatrix} - 2B_2 (t_1 \ \dots \ t_n) \underline{B}^{-1} \begin{Vmatrix} z_1 \\ \vdots \\ z_n \end{Vmatrix}$$

$$\left. \left. + n(\alpha_1^2 + \alpha_2^2) + \sum_{i=1}^{n} t_i^2 (B_1^2 + B_2^2) + 2n\bar{t} (\alpha_1 B_1 + \alpha_2 B_2) \right] \right\} \tag{3. 2. 6}$$

$$\times \quad dw_1 \dots dw_n \ dz_1 \dots dz_n ,$$

where $\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i$.

Since $\underline{B}$ is orthogonal we have

$$\underline{B}^{-1} = \underline{B}' = \left\| \begin{matrix} \frac{1}{\sqrt{n}} & \\ \frac{1}{\sqrt{n}} & \\ \vdots & \underline{C}' \\ \frac{1}{\sqrt{n}} & \end{matrix} \right\| \tag{3.2.7}$$

thus

$$(t_1 \; \ldots \; t_n) \, \underline{B}^{-1} = (\sqrt{n}\,\bar{t} \mid (t_1 \; \ldots \; t_n)\, \underline{C}') \, , \tag{3.2.8}$$

so (3.2.6) can be written as

$$M(T_o, T) = (2\pi)^{-n} \exp\left\{ -\frac{1}{2} \left[n(\alpha_1^{\,2} + \alpha_2^{\,2}) + (B_1^{\,2} + B_2^{\,2}) \sum_{i=1}^{n} t_i^{\,2} + 2n\bar{t}(\alpha_1 B_1 + \alpha_2 B_2) \right] \right\}$$

$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} \left[w_1^{\,2} + z_1^{\,2} - 2\alpha_1 \sqrt{n}\, w_1 - 2\alpha_2 \sqrt{n}\, z_1 \right.\right.$$

$$- 2 B_1 \sqrt{n}\,\bar{t}\, w_1 - 2 B_2 \sqrt{n}\,\bar{t}\, z_1 + (1-2T_o) \sum_{i=2}^{n} w_i^{\,2}$$

$$+ \sum_{i=2}^{n} z_i^{\,2} - 2 T \sum_{i=2}^{n} w_i z_i - 2B_1 (t_1 \; \ldots \; t_n)\, \underline{C}' \left\| \begin{matrix} w_2 \\ \vdots \\ w_n \end{matrix} \right\|$$

$$- 2B_2(t_1, \dots t_n)\ \underline{C}' \left\| \begin{matrix} z_2 \\ \vdots \\ z_n \end{matrix} \right\| \Big] \Big\} \ dw_1 \dots dw_n\, dz_1 \dots dz_n . \tag{3.2.9}$$

Integrating with respect to w_1 and z_1 we obtain

$$M(T_o, T) = (2\pi)^{-(n-1)}\, e^{-\frac{1}{2}\left[(B_1{}^2+B_2{}^2)\left(\sum_{i=1}^{n} t_i{}^2 - n\bar{t}^2\right)\right]}$$

$$\times \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\Big\{-\frac{1}{2}\Big[(1-2T_o)\sum_{i=2}^{n} w_i{}^2 + \sum_{i=2}^{n} z_i{}^2 - 2T\sum_{i=2}^{n} w_i z_i$$

$$- 2B_1(t_1 \dots t_n)\ \underline{C}' \left\| \begin{matrix} w_2 \\ \vdots \\ w_n \end{matrix} \right\| - 2B_2(t_1 \dots t_n)\ \underline{C}' \left\| \begin{matrix} z_2 \\ \vdots \\ z_n \end{matrix} \right\| \Big]\Big\}$$

$$\times \quad dw_2 \dots dw_n\ dz_2 \dots dz_n . \tag{3.2.10}$$

Except for a constant factor this is the product of n-1 double integrals of the form (3.1.5). Thus from (3.1.10) ,

$$M(T_o, T) = e^{-\frac{1}{2}(B_1{}^2+B_2{}^2)\sum_{i=1}^{n}(t_i - \bar{t})^2}$$

$$\times\ \left|\underline{A}\right|^{-\frac{(n-1)}{2}} \exp\left\{\sum_{i=2}^{n} \frac{1}{2}(a_i b_i)\ \underline{A}^{-1} \begin{pmatrix} a_i \\ b_i \end{pmatrix}\right\}, \tag{3.2.11}$$

where

$$\underline{A} = \begin{Vmatrix} 1-2\,T_o & -T \\ -T & 1 \end{Vmatrix} ,$$

$$(a_2 \; \ldots \; a_n) = B_1 (t_1 \; \ldots \; t_n) \, \underline{C'} \;, \quad \text{and}$$

$$(b_2 \; \ldots \; b_n) = B_2 (t_1 \; \ldots \; t_n) \, \underline{C'} \;.$$

Thus

$$M(T_o, T) = e^{-\frac{1}{2}(B_1{}^2 + B_2{}^2) \sum_{i=1}^{n} (t_i - \bar{t})^2}$$

$$\times \; (1-2T_o - T^2)^{-\frac{(n-1)}{2}} \exp\left\{ \frac{1}{2} \, \frac{\sum_{i=2}^{n} (a_i{}^2 + (1-2T_o) b_i{}^2 + 2T\, a_i b_i)}{(1-2T_o - T^2)} \right\} , \tag{3.2.12}$$

where

$$\frac{1}{B_1{}^2} \sum_{i=2}^{n} a_i{}^2 = \frac{1}{B_2{}^2} \sum_{i=2}^{n} b_i{}^2 = \frac{1}{B_1 B_2} \sum_{i=2}^{n} a_i \, b_i$$

$$= (t_1 \; \ldots \; t_n) \, \underline{C'} \, \underline{C} \begin{Vmatrix} t_1 \\ \vdots \\ t_n \end{Vmatrix} \tag{3.2.13}$$

Since $\underline{B}$ is orthogonal we have

$$\sum_{i=1}^{n} t_i^2 = (t_1 \ \dots \ t_n)\, \underline{B}'\, \underline{B} \begin{Vmatrix} t_1 \\ \vdots \\ t_n \end{Vmatrix} , \qquad (3.2.14)$$

thus from (3.2.8)

$$\sum_{i=1}^{n} t_i^2 = (\sqrt{n}\,\bar{t} \,|\, (t_1 \ \dots \ t_n)\, \underline{C}') \begin{Vmatrix} \sqrt{n}\,\bar{t} \\ \hline \underline{C} \begin{Vmatrix} t_1 \\ \vdots \\ t_n \end{Vmatrix} \end{Vmatrix} \qquad (3.2.15)$$

$$= n\,\bar{t}^2 + (t_1 \ \dots \ t_n)\, \underline{C}'\, \underline{C} \begin{Vmatrix} t_1 \\ \vdots \\ t_n \end{Vmatrix} ,$$

thus from (3.2.13)

$$\frac{1}{B_1^2} \sum_{i=2}^{n} a_i^2 = \frac{1}{B_2^2} \sum_{i=2}^{n} b_i^2 = \frac{1}{B_1 B_2} \sum_{i=2}^{n} a_i b_i = \sum_{i=1}^{n} (t_i - \bar{t})^2 . \qquad (3.2.16)$$

Substituting (3.2.16) in (3.2.12) we obtain

$$M(T_o, T) = e^{-\frac{1}{2}(B_1^2 + B_2^2) \sum_{i=1}^{n} (t_i - \bar{t})^2} \qquad (3.2.17)$$

$$\times\ (1-2T_o-T^2)^{-\frac{(n-1)}{2}} \exp\left\{\frac{\sum_{i=1}^{n}(t_i-\bar{t})^2}{2(1-2T_o-T^2)}\left[B_1{}^2+(1-2T_o)B_2{}^2+2TB_1B_2\right]\right\}.$$

We see that (3. 2. 17) can be obtained from (3. 1. 12) by replacing

$$\sum_{i=1}^{n} t_i{}^2 \text{ and } (1-2T_o-T^2)^{-\frac{n}{2}} \quad \text{by}$$

$$\sum_{i=1}^{n} (t_i-\bar{t})^2 \text{ and } (1-2T_o-T^2)^{-\frac{(n-1)}{2}} \quad \text{respectively.}$$

Thus, if we follow the same procedure as in Section I from (3. 1. 12) onwards, with

$$m-1 = \sum_{i=1}^{n} (t_i-\bar{t})^2 ,$$

and assume

$$m = O(n) ,$$

we obtain the expression corresponding to (3. 1.29) for the p. d. f. of $b_{21}{}^{x}$,

$$h(b_{21}{}^{x}) = \sqrt{\frac{n-1}{2\pi}}\ e^{-\frac{(n-1)}{2}(B_1{}^2+B_2{}^2)} \quad (1+b_{21}{}^{x2})^{-\frac{n}{2}} \tag{3.2.18}$$

$$\times \frac{\left[1 + A^2 + 2k\hat{z} + (B_2^{\,2}-1)\hat{z}^2\right]}{(1-\hat{z}^2)^{\frac{n-1}{2}+1}\left[p^2+1+2k\hat{z}-3\hat{z}^2\right]^{\frac{1}{2}}} \quad \exp\left\{\frac{n-1}{2}\left[\frac{2k\hat{z}+p^2}{(1-\hat{z}^2)}\right]\right\}$$

$$\times \left\{1 + O(n^{-1})\right\} \qquad (n \to \infty),$$

where A, k, p^2 are defined by (3. 1. 18) with b_{21} replaced by b_{21}^{x}.

A Fortran program was written to compile re-normalised tables of the approximate distribution of b_{21} given in (3. 1. 29). It can be seen from (3. 2. 18) that the distribution of b_{21}^{x} for n degrees of freedom is the same as that of b_{21} for n-1 degrees of freedom. Also from the distribution of b_{21} (3. 1. 29), definitions (3. 1. 18), and the fact that $\hat{z}$ is a root of (3. 1. 25) it can be seen that the following pairs of values for B_1, B_2 and n give identical distributions.

(i) $B_1 = b_1$, $B_2 = b_2$, n, and $B_1 = -b_1$, $B_2 = -b_2$, n;

(ii) $B_1 = 0$, $B_2 = b$, n, and $B_1 = 0$, $B_2 = -b$, n;

(iii) $B_1 = b$, $B_2 = 0$, n and $B_1 = -b$, $B_2 = 0$, n;

and that the distribution is symmetric if either $B_1 = 0$ or $B_2 = 0$. Also the values $B_1 = -b_1$, $B_2 = b_2$, n, give a distribution which is a reflection in the axis of b_{21} of the distribution given by $B_1 = -b$, $B_2 = b_2$, n.

The tables in Appendix II show the 5 distinct distributions of

b_{21} where the values for B_1 and B_2 are chosen from the integers -1, 0, 1, for each of the values 15, 30, 45, for n. It can be seen that in all the cases a measure of central tendency is the value

$$\frac{B_1 B_2}{1+B_1^2} = \frac{E \sum_{i=1}^{n} x_i y_i}{E \sum_{i=1}^{n} x_i^2}$$

in the case (3.1.2), and this value is the mean in the cases where the distribution is symmetric. In each case the variance of b_{21} decreases approximately as $\frac{1}{n}$. For each value of n the variance for the case $B_1 = 0$, $B_2 = 1$ is approximately twice that for the case $B_1 = 0$, $B_2 = 0$, which latter is approximately twice that for the remaining cases $B_1 = 1$, $B_2 = 0$; $B_1 = 1$, $B_2 = 1$; $B_1 = -1$, $B_2 = 1$.

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APPENDIX I

DANIELS'S FORM OF GEARY'S (1944) EXTENSION OF CRAMER'S THEOREM

We are interested in the distribution of statistics of the form $r = \frac{c}{c_o}$, where c_o is non-negative. If c_o, c have a joint p.d.f. $f(c_o, c)$, the density for r is

$$h(r) = \int_0^\infty c_o \, f(c_o, \; r\,c_o)\, d\,c_o \, .$$

Let

$$M(T_o, T) = E\, e^{T_o c_o + Tc}$$

be the joint moment-generating function for c_o and c. We are concerned only with cases where $M(T_o, T)$ exists in strips of non-zero width containing the imaginary axes in the T_o and T planes. The usual Fourier inversion formula is most conveniently written as

$$f(c_o, c) = \frac{1}{(2\pi i)^2} \iint M(T_o, T)\, e^{-T_o c_o - Tc} \, dT_o dT \, ,$$

where integration is along the imaginary axes of T_o and T, or any allowable deformations of these paths. In particular,

$$f(c_o, \; rc_o) = \frac{1}{(2\pi i)^2} \iint M(T_o, T)\, e^{-(T_o + rT)c_o} \, d\,T_o \, dT$$

$$= \frac{1}{(2\pi i)^2} \iint M(U - rT, T)\, e^{-u\, c_o} \, d\,u\, d\,T \, ,$$

where the integration of $u = T_o + rT$ is taken over a similar path in the u plane. Inversion of the transform with respect to u gives

$$\int_0^\infty f(c_o, r c_o) e^{u c_o} d c_o$$

$$= \frac{1}{2\pi i} \int M(u - r T, T) dT ,$$

so that, when differentiation is permissible,

$$\int_0^\infty c_o f(c_o, r c_o) e^{u c_o} d c_o$$

$$= \frac{1}{2\pi i} \int \frac{\partial M(u - r T, T)}{\partial u} dT$$

and

$$h(r) = \frac{1}{2\pi i} \int \left. \frac{\partial M(u - rT, T)}{\partial u} \right|_{u=0} dT .$$

APPENDIX II

EXAMPLE TABLES OF THE APPROXIMATE DISTRIBUTION OF THE SAMPLE REGRESSION COEFFICIENT

The following tables provide examples of the renormalized density and distribution functions of the approximate distribution of b_{21}. Thus if $h(b_{21})$ is defined by equation (3. 1. 29), the columns in the table give values of $h(b_{21})/\text{NORM}$, $H(b_{21})/\text{NORM}$ and $\hat{z}$ for various values of b_{21} centred at $B_1 B_2/(1+B_1{}^2)$, where

$$\text{NORM} = \int_{-\infty}^{\infty} h(b_{21})\, d\, b_{21}, \quad H(b_{21}) = \int_{-\infty}^{\infty} h(x)dx$$

and $\hat{z}$ is defined in section 3. 1. The mean and variance of the renormalized distribution and the value NORM of the normalizing constant are given at the bottom of each table.

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$B_1 = 0.0$ $B_2 = 1.0$ N = 15

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-2.200	-.00004	.00001	.39298
-2.100	.00007	.00001	.39096
-2.000	.00013	.00002	.38866
-1.900	.00022	.00004	.38601
-1.800	.00039	.00007	.38294
-1.700	.00070	.00012	.37937
-1.600	.00125	.00022	.37519
-1.500	.00225	.00039	.37027
-1.400	.00406	.00070	.36445
-1.300	.00732	.00126	.35751
-1.200	.01314	.00226	.34920
-1.100	.02336	.00405	.33919
-1.000	.04094	.00721	.32711
-.900	.07025	.01268	.31248
-.800	.11719	.02192	.29478
-.700	.18862	.03703	.27343
-.600	.29057	.06079	.24786
-.500	.42513	.09638	.21759
-.400	.58648	.14683	.18232
-.300	.75817	.21407	.14214
-.200	.91398	.29787	.09758
-.100	1.02386	.39514	.04969
0.000	1.06359	.50000	0.00000
.100	1.02386	.60486	-.04969
.200	.91398	.70213	-.09758
.300	.75817	.78593	-.14214
.400	.58648	.85317	-.18232
.500	.42513	.90362	-.21759
.600	.29057	.93921	-.24786
.700	.18862	.96297	-.27343
.800	.11719	.97808	-.29478
.900	.07025	.98732	-.31248
1.000	.04094	.99279	-.32711
1.100	.02336	.99595	-.33919
1.200	.01314	.99774	-.34920
1.300	.00732	.99874	-.35751
1.400	.00406	.99930	-.36445
1.500	.00225	.99961	-.37027
1.600	.00125	.99978	-.37519
1.700	.00070	.99988	-.37937
1.800	.00039	.99993	-.38294
1.900	.00022	.99996	-.38601
2.000	.00013	.99998	-.38866
2.100	.00007	.99999	-.39096
2.200	.00004	.99999	-.39298

MEAN = 0.00000 VARIANCE = .15327 NORM = 1.02723

$B_1 = 0.0$ $\qquad B_2 = 0.0$ $\qquad N = 15$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-2.200	.00000	.00000	0.00000
-2.100	.00000	.00000	0.00000
-2.000	.00000	.00000	0.00000
-1.900	.00001	.00000	0.00000
-1.800	.00001	.00000	0.00000
-1.700	.00003	.00000	0.00000
-1.600	.00006	.00001	0.00000
-1.500	.00012	.00002	0.00000
-1.400	.00026	.00004	0.00000
-1.300	.00055	.00007	0.00000
-1.200	.00121	.00016	0.00000
-1.100	.00267	.00035	0.00000
-1.000	.00594	.00076	0.00000
-.900	.01319	.00168	0.00000
-.800	.02904	.00372	0.00000
-.700	.06255	.00815	0.00000
-.600	.12984	.01749	0.00000
-.500	.25494	.03628	0.00000
-.400	.46351	.07162	0.00000
-.300	.76262	.13242	0.00000
-.200	1.11034	.22602	0.00000
-.100	1.40330	.35245	0.00000
0.000	1.51958	.50000	0.00000
.100	1.40330	.64755	0.00000
.200	1.11034	.77398	0.00000
.300	.76262	.86758	0.00000
.400	.46351	.92838	0.00000
.500	.25494	.96372	0.00000
.600	.12984	.98251	0.00000
.700	.06255	.99185	0.00000
.800	.02904	.99628	0.00000
.900	.01319	.99832	0.00000
1.000	.00594	.99924	0.00000
1.100	.00267	.99965	0.00000
1.200	.00121	.99984	0.00000
1.300	.00055	.99993	0.00000
1.400	.00026	.99996	0.00000
1.500	.00012	.99998	0.00000
1.600	.00006	.99999	0.00000
1.700	.00003	1.00000	0.00000
1.800	.00001	1.00000	0.00000
1.900	.00001	1.00000	0.00000
2.000	.00000	1.00000	0.00000
2.100	.00000	1.00000	0.00000
2.200	.00000	1.00000	0.00000

MEAN= .00000 VARIANCE= .07692 NORM= 1.01679

$B_1 = 1.0$ $B_2 = 0.0$ $N = 15$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.500	.00000	.00000	0.00000
-1.400	.00000	.00000	0.00000
-1.300	.00001	.00000	0.00000
-1.200	.00002	.00000	0.00000
-1.100	.00005	.00000	0.00000
-1.000	.00017	.00002	0.00000
-.900	.00058	.00005	0.00000
-.800	.00198	.00017	0.00000
-.700	.00689	.00057	0.00000
-.600	.02358	.00198	0.00000
-.500	.07662	.00662	0.00000
-.400	.22567	.02086	0.00000
-.300	.57070	.05912	0.00000
-.200	1.16993	.14473	0.00000
-.100	1.84512	.29623	0.00000
0.000	2.15736	.50000	0.00000
.100	1.84512	.70377	0.00000
.200	1.16993	.85527	0.00000
.300	.57070	.94088	0.00000
.400	.22567	.97914	0.00000
.500	.07662	.99338	0.00000
.600	.02358	.99802	0.00000
.700	.00689	.99943	0.00000
.800	.00198	.99983	0.00000
.900	.00058	.99995	0.00000
1.000	.00017	.99998	0.00000
1.100	.00005	1.00000	0.00000
1.200	.00002	1.00000	0.00000
1.300	.00001	1.00000	0.00000
1.400	.00000	1.00000	0.00000
1.500	.00000	1.00000	0.00000

MEAN= .00000 VARIANCE= .03715 NORM= 1.01286

$B_1 = 1.0$ $B_2 = 1.0$ $N = 15$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.150	.00000	.00000	.04915
-1.050	.00000	.00000	.01724
-.950	.00000	.00000	-.01812
-.850	.00001	.00000	-.05705
-.750	.00002	.00000	-.09949
-.650	.00005	.00001	-.14510
-.550	.00014	.00001	-.19309
-.450	.00041	.00004	-.24212
-.350	.00128	.00012	-.29019
-.250	.00419	.00037	-.33471
-.150	.01419	.00121	-.37293
-.050	.04750	.00406	-.40273
.050	.14787	.01319	-.42349
.150	.39832	.03926	-.43625
.250	.86894	.10125	-.44307
.350	1.46669	.21803	-.44614
.450	1.88258	.38773	-.44713
.550	1.85500	.57761	-.44715
.650	1.44420	.74418	-.44684
.750	.92379	.86241	-.44649
.850	.50617	.93293	-.44624
.950	.24706	.96966	-.44613
1.050	.11114	.98697	-.44612
1.150	.04738	.99458	-.44621
1.250	.01957	.99778	-.44635
1.350	.00797	.99909	-.44652
1.450	.00324	.99963	-.44669
1.550	.00132	.99985	-.44686
1.650	.00055	.99993	-.44699
1.750	.00023	.99997	-.44710
1.850	.00010	.99999	-.44717
1.950	.00004	.99999	-.44721
2.050	.00002	1.00000	-.44721
2.150	.00001	1.00000	-.44717
2.250	.00000	1.00000	-.44711
2.350	.00000	1.00000	-.44701
2.450	.00000	1.00000	-.44688

MEAN= .51702 VARIANCE= .04753 NORM= 1.01332

$B_1 = -1.0$ $B_2 = 1.0$ $N = 15$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-2.450	.00000	.00000	.44688
-2.350	.00000	.00000	.44701
-2.250	.00000	.00000	.44711
-2.150	.00001	.00000	.44717
-2.050	.00002	.00000	.44721
-1.950	.00004	.00001	.44721
-1.850	.00010	.00001	.44717
-1.750	.00023	.00003	.44710
-1.650	.00055	.00007	.44699
-1.550	.00132	.00015	.44686
-1.450	.00324	.00037	.44669
-1.350	.00797	.00091	.44652
-1.250	.01957	.00222	.44635
-1.150	.04738	.00542	.44621
-1.050	.11114	.01303	.44612
-.950	.24706	.03034	.44613
-.850	.50617	.06707	.44624
-.750	.92379	.13759	.44649
-.650	1.44420	.25582	.44684
-.550	1.85500	.42239	.44715
-.450	1.88258	.61227	.44713
-.350	1.46669	.78197	.44614
-.250	.86894	.89875	.44307
-.150	.39832	.96074	.43625
-.050	.14787	.98681	.42349
.050	.04750	.99594	.40273
.150	.01419	.99879	.37293
.250	.00419	.99963	.33471
.350	.00128	.99988	.29019
.450	.00041	.99996	.24212
.550	.00014	.99999	.19309
.650	.00005	.99999	.14510
.750	.00002	1.00000	.09949
.850	.00001	1.00000	.05705
.950	.00000	1.00000	.01812
1.050	.00000	1.00000	-.01724
1.150	.00000	1.00000	-.04915

MEAN= -.51702 VARIANCE= .04753 NORM= 1.01332

$B_1 = 0.0$ $B_2 = 1.0$ $N = 30$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.700	.00000	.00000	.37937
-1.600	.00000	.00000	.37519
-1.500	.00001	.00000	.37027
-1.400	.00003	.00000	.36445
-1.300	.00009	.00001	.35751
-1.200	.00028	.00002	.34920
-1.100	.00085	.00008	.33919
-1.000	.00255	.00024	.32711
-.900	.00730	.00070	.31248
-.800	.01986	.00199	.29478
-.700	.05045	.00534	.27343
-.600	.11779	.01345	.24786
-.500	.24894	.03129	.21759
-.400	.46936	.06660	.18232
-.300	.77944	.12857	.14214
-.200	1.12828	.22399	.09758
-.100	1.41292	.35182	.04969
0.000	1.52373	.50000	0.00000
.100	1.41292	.64818	-.04969
.200	1.12828	.77601	-.09758
.300	.77944	.87143	-.14214
.400	.46936	.93340	-.18232
.500	.24894	.96871	-.21759
.600	.11779	.98655	-.24786
.700	.05045	.99466	-.27343
.800	.01986	.99801	-.29478
.900	.00730	.99930	-.31248
1.000	.00255	.99976	-.32711
1.100	.00085	.99992	-.33919
1.200	.00028	.99998	-.34920
1.300	.00009	.99999	-.35751
1.400	.00003	1.00000	-.36445
1.500	.00001	1.00000	-.37027
1.600	.00000	1.00000	-.37519
1.700	.00000	1.00000	-.37937

MEAN= .00000 VARIANCE= .07134 NORM= 1.01403

$B_1 = 0.0$ $B_2 = 0.0$ $N = 30$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.300	.00000	.00000	0.00000
-1.200	.00000	.00000	0.00000
-1.100	.00001	.00000	0.00000
-1.000	.00005	.00000	0.00000
-.900	.00022	.00001	0.00000
-.800	.00101	.00007	0.00000
-.700	.00448	.00031	0.00000
-.600	.01845	.00135	0.00000
-.500	.06819	.00532	0.00000
-.400	.21715	.01866	0.00000
-.300	.56983	.05641	0.00000
-.200	1.17987	.14249	0.00000
-.100	1.85725	.29516	0.00000
0.000	2.16697	.50000	0.00000
.100	1.85725	.70484	0.00000
.200	1.17987	.85751	0.00000
.300	.56983	.94359	0.00000
.400	.21715	.98134	0.00000
.500	.06819	.99468	0.00000
.600	.01845	.99865	0.00000
.700	.00448	.99969	0.00000
.800	.00101	.99993	0.00000
.900	.00022	.99999	0.00000
1.000	.00005	1.00000	0.00000
1.100	.00001	1.00000	0.00000
1.200	.00000	1.00000	0.00000
1.300	.00000	1.00000	0.00000

MEAN= .00000 VARIANCE= .03571 NORM= 1.00837

$B_1 = 1.0$ $B_2 = 0.0$ $N = 30$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-.900	.00000	.00000	0.00000
-.800	.00000	.00000	0.00000
-.700	.00004	.00000	0.00000
-.600	.00046	.00002	0.00000
-.500	.00456	.00022	0.00000
-.400	.03750	.00195	0.00000
-.300	.22913	.01346	0.00000
-.200	.92994	.06689	0.00000
-.100	2.26299	.22446	0.00000
0.000	3.07073	.50000	0.00000
.100	2.26299	.77554	0.00000
.200	.92994	.93311	0.00000
.300	.22913	.98654	0.00000
.400	.03750	.99805	0.00000
.500	.00456	.99978	0.00000
.600	.00046	.99998	0.00000
.700	.00004	1.00000	0.00000
.800	.00000	1.00000	0.00000
.900	.00000	1.00000	0.00000

MEAN= .00000 VARIANCE= .01756 NORM= 1.00634

$B_1 = 1.0$ $B_2 = 1.0$ $N = 30$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-.350	.00000	.00000	-.29019
-.250	.00002	.00000	-.33471
-.150	.00020	.00001	-.37293
-.050	.00200	.00010	-.40273
.050	.01791	.00090	-.42349
.150	.12262	.00686	-.43625
.250	.56241	.03788	-.44307
.350	1.57602	.14159	-.44614
.450	2.59970	.35414	-.44713
.550	2.56448	.62054	-.44715
.650	1.59762	.83131	-.44684
.750	.67777	.94251	-.44649
.850	.21233	.98436	-.44624
.950	.05302	.99643	-.44613
1.050	.01128	.99928	-.44612
1.150	.00216	.99987	-.44621
1.250	.00039	.99998	-.44635
1.350	.00007	1.00000	-.44652
1.450	.00001	1.00000	-.44669
1.550	.00000	1.00000	-.44686

MEAN= .50842 VARIANCE= .02218 NORM= 1.00655

$B_1 = -1.0$ $B_2 = 1.0$ $N = 30$

b_{21}	$h(b_{21})/NORM$	$H(b_{21})/NORM$	z
-1.550	.00000	.00000	.44686
-1.450	.00001	.00000	.44669
-1.350	.00007	.00000	.44652
-1.250	.00039	.00002	.44635
-1.150	.00216	.00013	.44621
-1.050	.01128	.00072	.44612
-.950	.05302	.00357	.44613
-.850	.21233	.01564	.44624
-.750	.67777	.05749	.44649
-.650	1.59762	.16869	.44684
-.550	2.56448	.37946	.44715
-.450	2.59970	.64586	.44713
-.350	1.57602	.85841	.44614
-.250	.56241	.96212	.44307
-.150	.12262	.99314	.43625
-.050	.01791	.99910	.42349
.050	.00200	.99990	.40273
.150	.00020	.99999	.37293
.250	.00002	1.00000	.33471
.350	.00000	1.00000	.29019

MEAN= -.50842 VARIANCE= .02218 NORM= 1.0065

$B_1 = 0.0$ $B_2 = 1.0$ $N = 45$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.300	.00000	.00000	.35751
-1.200	.00001	.00000	.34920
-1.100	.00003	.00000	.33919
-1.000	.00014	.00001	.32711
-.900	.00065	.00004	.31248
-.800	.00289	.00020	.29478
-.700	.01159	.00086	.27343
-.600	.04100	.00328	.24786
-.500	.12518	.01109	.21759
-.400	.32257	.03256	.18232
-.300	.68812	.08197	.14214
-.200	1.19609	.17565	.09758
-.100	1.67443	.32016	.04969
0.000	1.87461	.50000	0.00000
.100	1.67443	.67984	-.04969
.200	1.19609	.82435	-.09758
.300	.68812	.91803	-.14214
.400	.32257	.96744	-.18232
.500	.12518	.98891	-.21759
.600	.04100	.99672	-.24786
.700	.01159	.99914	-.27343
.800	.00289	.99980	-.29478
.900	.00065	.99996	-.31248
1.000	.00014	.99999	-.32711
1.100	.00003	1.00000	-.33919
1.200	.00001	1.00000	-.34920
1.300	.00000	1.00000	-.35751

MEAN= .00000 VARIANCE= .04648 NORM= 1.00[illegible]

$B_1 = 0.0$ $B_2 = 0.0$ $N = 45$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.000	.00000	.00000	0.00000
-.900	.00000	.00000	0.00000
-.800	.00003	.00000	0.00000
-.700	.00028	.00001	0.00000
-.600	.00226	.00012	0.00000
-.500	.01571	.00087	0.00000
-.400	.08761	.00537	0.00000
-.300	.36669	.02605	0.00000
-.200	1.07978	.09522	0.00000
-.100	2.11696	.25490	0.00000
0.000	2.66136	.50000	0.00000
.100	2.11696	.74510	0.00000
.200	1.07978	.90478	0.00000
.300	.36669	.97395	0.00000
.400	.08761	.99463	0.00000
.500	.01571	.99913	0.00000
.600	.00226	.99988	0.00000
.700	.00028	.99999	0.00000
.800	.00003	1.00000	0.00000
.900	.00000	1.00000	0.00000
1.000	.00000	1.00000	0.00000

MEAN= .00000 VARIANCE= .02326 NORM= 1.00557

$B_1 = 1.0$ $B_2 = 0.0$ $N = 45$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-.700	.00000	.00000	0.00000
-.600	.00001	.00000	0.00000
-.500	.00023	.00001	0.00000
-.400	.00537	.00021	0.00000
-.300	.07933	.00343	0.00000
-.200	.63737	.03360	0.00000
-.100	2.39326	.17784	0.00000
0.000	3.76885	.50000	0.00000
.100	2.39326	.82216	0.00000
.200	.63737	.96640	0.00000
.300	.07933	.99657	0.00000
.400	.00537	.99979	0.00000
.500	.00023	.99999	0.00000
.600	.00001	1.00000	0.00000
.700	.00000	1.00000	0.00000

MEAN= .00000 VARIANCE= .01150 NORM= 1.00421

$B_1 = 1.0$ $B_2 = 1.0$ $N = 45$

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-.150	.00000	.00000	-.37293
-.050	.00007	.00000	-.40273
.050	.00187	.00007	-.42349
.150	.03254	.00135	-.43625
.250	.31383	.01550	-.44307
.350	1.46003	.09713	-.44614
.450	3.09505	.32830	-.44713
.550	3.05653	.65061	-.44715
.650	1.52368	.88188	-.44684
.750	.42872	.97377	-.44649
.850	.07679	.99594	-.44624
.950	.00981	.99953	-.44613
1.050	.00099	.99995	-.44612
1.150	.00008	1.00000	-.44621
1.250	.00001	1.00000	-.44635
1.350	.00000	1.00000	-.44652

MEAN= .50560 VARIANCE= .01447 NORM= 1.0043'

$B_1 = -1.0$ $B_2 = 1.0$ N = 45

b_{21}	$h(b_{21})$/NORM	$H(b_{21})$/NORM	z
-1.350	.00000	.00000	.44652
-1.250	.00001	.00000	.44635
-1.150	.00008	.00000	.44621
-1.050	.00099	.00005	.44612
-.950	.00981	.00047	.44613
-.850	.07679	.00406	.44624
-.750	.42872	.02623	.44649
-.650	1.52368	.11812	.44684
-.550	3.05653	.34939	.44715
-.450	3.09505	.67170	.44713
-.350	1.46003	.90287	.44614
-.250	.31383	.98450	.44307
-.150	.03254	.99865	.43625
-.050	.00187	.99993	.42349
.050	.00007	1.00000	.40273
.150	.00000	1.00000	.37293

MEAN= -.50560 VARIANCE= .01447 NORM= 1.0043